

Large spin expansion of semiclassical 3-point correlators in $AdS_5 \times S^5$

J.G. Russo^{a,b} and A.A. Tseytlin^{c*}

^a Institució Catalana de Recerca i Estudis Avançats (ICREA)
Pg. Lluís Companys, 23, 08010 Barcelona, Spain.

^b Institute of Cosmos Sciences and ECM, Facultat de Física,
Universitat de Barcelona, Diagonal 647, 08028 Barcelona, Spain.

^c Blackett Laboratory, Imperial College, London SW7 2AZ, U.K.

Abstract

3-point correlators in $AdS_5 \times S^5$ string theory in which two states are “heavy” (have large quantum numbers) and the third is “light” (here chosen as chiral primary scalar) can be computed semiclassically in terms of the “light” vertex operator evaluated on the classical string solution sourced by the two “heavy” operators. We observe that in the case when the “heavy” operators represent BPS states there is an ambiguity in the computation depending on whether the mass shell (or marginality) condition is imposed before or after integration over the world sheet. We show that this ambiguity is resolved in a universal way by defining the BPS correlator as a limit of the one with non-BPS “heavy” states. We consider several examples with “heavy” states represented by folded or circular spinning strings in $AdS_5 \times S^5$ that admit a point-like BMN-type limit when one S^5 spin J is much larger than the others. Remarkably, in all of these cases the large J expansion of the 3-point correlator has the same structure as expected in perturbative (tree-level and one-loop) dual gauge theory. We conjecture that, like the leading chiral primary correlator term, the coefficients of the first few subleading terms are also protected, i.e. should be the same at strong and weak coupling.

*Also at Lebedev Institute, Moscow.

1 Introduction

A complete proof of (planar) AdS/CFT duality requires matching not only 2-point but also 3-point correlation functions of primary operators as they determine the structure of a CFT. During the last few years an impressive progress was achieved in understanding the integrability-based equations determining dimensions of general non-BPS operators (and thus their 2-point functions) at any value of gauge coupling or string tension (see, e.g., [1]). However, the properties of tree-level string and planar gauge theory 3-point functions remain largely an unexplored area.

Recently, there was a renewed interest in 2-point and 3-point correlators in $AdS_5 \times S^5$ string theory in the semiclassical limit of large quantum numbers (see, e.g., [2–8]). In the case when two of the three string vertex operators represent “heavy” states that carry large charges while the third corresponds to a “light” BPS state with fixed charge it was suggested in [5, 6] that such 3-point correlator can be computed by evaluating the “light” operator on the classical solution “sourced” [9] by the “heavy” operators. This observation was further generalized and extended to the case when the “light” operator may represent a non-BPS string state in [7].

Such semiclassical method of computing 3-point functions involving two “heavy” and one “light” state can be explicitly tested in string theory in flat space-time [10]. Such correlator describes, e.g., a decay of rotating massive string by radiation emission. For example, for a specific state representing a circular string rotating in several planes, one can independently carry out an exact (tree-level) quantum string calculation in terms of creation/annihilation operators, producing a rather non-trivial expression in terms of an infinite sum. In the limit of large mass of the “heavy” state, in which the semiclassical approach should be reliable, the semiclassical calculation reproduces *exactly* the result of the quantum string calculation.

Here we shall follow closely the approach of [7]. Our aim will be to analyse the large angular momentum expansion of the semiclassical expressions for correlation functions of two “heavy” non-BPS operators and one “light” chiral primary operator.

It was noticed in [5] that the BMN-type limit ($J_1 \gg J_2$) of such 3-point function involving a “heavy” state corresponding to the folded spinning (J_1, J_2) string on S^5 contains an extra term as compared to the correlator of the three BPS (chiral primary) operators computed directly. This extra term may be interpreted as arising from a “ $\frac{0}{0}$ ”-type ambiguity. Here we shall clarify the universal nature of this “anomalous” contribution showing that it always appears for generic string states that admit a point-like BMN limit. We shall see that to have a smooth BPS limit of a non-BPS correlator one may formally adjust the normalization of the “light” chiral primary operator.

We will also study the subleading terms in the large spin expansion of the semiclassical 3-point correlator. In general, the semiclassical string expansion is based on taking string tension large, i.e. $\sqrt{\lambda} \gg 1$, while fixing the classical spin parameters $\mathcal{J}_i = \frac{J_i}{\sqrt{\lambda}}$. While this limit is, of course, different from the large J limit in perturbative gauge theory (where one first expands in $\lambda \ll 1$ and then takes J_i large) some leading terms in the string and gauge theory observables expanded at large J may happen to have the same structure. Moreover, the coefficients of these terms may happen to be protected (due to supersymmetry), i.e. may be the same at strong and weak coupling. This is, indeed, what happens, e.g., in the expansion of the energy of a

classical string on S^5 with two spins in the limit $J_1 \gg J_2$ when the c.o.m. orbital momentum is large [11]: $E = J + c_1 \frac{\lambda}{J} + c_2 \frac{\lambda^2}{J^3} + \dots$, where $J = J_1 + J_2$ and c_i are functions of $\frac{J_2}{J}$ (one may then further expand in $\frac{J_2}{J} \ll 1$ which would correspond to approaching the BMN limit). Not only this expansion has the same form as the corresponding large J expansion in the perturbative gauge theory, but also the first few leading coefficients c_i match exactly, i.e. are protected against λ -dependent corrections [11–16] (for a review, see, e.g., [17]).

Since the one-loop corrections to 3-point correlators in dual gauge theory often have structure similar to one-loop anomalous dimensions of the operators involved (see, e.g., [18, 19]) one may wonder if the leading terms in the large spin expansion of the semiclassical string-theory correlators may also follow the pattern observed in the expansion of semiclassical string energies. Indeed, the coefficients in the semiclassical string 3-point correlators can be expanded in a similar way in powers of $\frac{1}{J} = \frac{\sqrt{\lambda}}{J}$ for fixed $\frac{J_2}{J}$. In addition, one may further expand in $\frac{J_2}{J} \ll 1$, i.e. consider a near BMN limit. As we shall demonstrate below on several examples, the large J expansion of the semiclassical string 3-point correlators has formally the same structure as would be expected in gauge theory. We are then led to conjecture that, like the leading (three chiral primary correlator) term, the coefficients of the first few subleading terms in this expansion may also be protected, i.e. may turn out to be the same on the perturbative string and the perturbative gauge theory sides. It would be very interesting to check this conjecture by direct computation of the corresponding correlators on the weak coupling gauge theory side. This should hopefully become possible soon using the integrability-based approach suggested in [20].

The structure of this paper is as follows. In section 2 we review the basic setup for the semiclassical calculation of the correlators in $AdS_5 \times S^5$ involving two “heavy” and one “light” (chiral primary) state and discuss the ambiguity that arises in the case when all three operators are BPS (chiral primary) ones and suggest its resolution. As a test of our proposal, in section 3 we show that in the case when the “heavy” states are represented by rigid strings of arbitrary shape rotating on S^5 the “anomalous” rescaling of the correlator in the large spin limit can be removed by adjusting the normalization of the “light” chiral primary vertex operator.

To study subleading terms in the large spin expansion of the semiclassical correlator in section 4 we consider few explicit examples: folded string with two angular momenta extended on S^5 , folded string extended on AdS_5 with rotation in both AdS_5 and S^5 , circular strings extended and rotating in both AdS_5 and S^5 . We show that the large S^5 spin expansion can be organised as expansion in powers of $\frac{\lambda}{J^2}$, i.e. has the same structure as expected in perturbative gauge theory.

2 Ambiguity in semiclassical computation of correlator of three chiral primary operators

Let us start with revisiting the computation of the correlator of three chiral primary operators with charges $J_1 \approx J_2 \equiv J \gg J_3 \equiv j$. Following [5, 7] the coefficient in the 3-point function can be found by evaluating the vertex operator corresponding to the “light” state with $SO(6)$ orbital momentum j on the classical solution “sourced” by the two “heavy” BMN operators

with charges $J, -J$.¹

The relevant part of the integrated CPO vertex operator [5, 21] inserted at point $x^m = 0$ at the boundary of the Poincaré patch of AdS_5 ($ds^2 = z^{-2}(dz^2 + dx^m dx_m)$) is [7]²

$$V_{\Delta}^{(cpo)}(0) = \hat{c}_j \int_{-\infty}^{\infty} d\tau_e \int_0^{2\pi} d\sigma (Y_+)^{-\Delta} X_x^j U^{(cpo)}(x, z, X) , \quad \Delta = j , \quad (2.1)$$

$$U^{(cpo)} = \frac{\sqrt{\lambda}}{2\pi} \left[z^{-2} (\partial^a x^m \partial_a x_m - \partial^a z \partial_a z) - \partial^a X_k \partial_a X_k \right] , \quad \hat{c}_j = \frac{\sqrt{j}(j+1)}{4N} . \quad (2.2)$$

Here

$$Y_+ = Y_5 + Y_4 = z + z^{-1} x^m x_m , \quad X_x = X_1 + iX_2 = \cos \vartheta e^{i\varphi} , \quad (2.3)$$

where Y_M and X_k are embedding coordinates of AdS_5 and S^5 and c_j is a normalization constant of the chiral primary vertex operator that will be discussed later. For a “heavy”-state classical solution satisfying $z^2 + x^m x_m = 1$ (all solutions discussed below will be of that type) one then finds the 3-point correlator coefficients as [7]³

$$\begin{aligned} C_{123} &= \frac{\langle V_H(x) V_H(-x) V_{\Delta}^{(cpo)}(0) \rangle}{\langle V_H(x) V_H(-x) \rangle} \\ &= c_j \left\{ \int_{-\infty}^{\infty} d\tau_e \int_0^{2\pi} d\sigma z^j X_x^j U^{(cpo)}(x, z, X) \right\}_{class} , \quad c_j = 2^{-j} \hat{c}_j , \end{aligned} \quad (2.4)$$

where we assumed that $|x| = 1$ and the r.h.s. is evaluated on the classical solution determined by the “heavy” state.

The classical solution ($t = \kappa\tau$, $\varphi = \nu\tau$, $\kappa = \nu$) describing a point-like string with large orbital momentum J in S^5 is a massive geodesic in AdS_5 , running through its center. Written in the Poincaré coordinates, after a euclidean continuation it is

$$z = \frac{1}{\cosh(\kappa\tau_e)} , \quad x_{0e} = \tanh(\kappa\tau_e) , \quad x_i = 0 , \quad (2.5)$$

$$\varphi = -i\nu\tau_e , \quad \kappa^2 = \nu^2 , \quad J = \sqrt{\lambda} \nu . \quad (2.6)$$

The radial coordinate z vanishes in the limits $\tau_e \rightarrow \pm\infty$, implying that the euclidean trajectory reaches the boundary at the two points: $x_{0e} = -1$, $x_i = 0$ and $x_{0e} = 1$, $x_i = 0$. Evaluating (2.4) on this solution, but without imposing the “on-shell” (or Virasoro) condition $\kappa^2 = \nu^2$, we

¹Below we shall assume that $J > 0$, $j > 0$. The charge conservation requires that the operators have charges $J, -J - j, j$ but for $J \gg j$ one has $-J - j \approx -J$.

²We shall follow closely the notation in [7] where details may be found. Here we consider the Euclidean continuation: $\tau_e = i\tau$. Compared to [7] here we extract the factor of string tension $T = \frac{\sqrt{\lambda}}{2\pi}$ from c_j . N here stands for a factor of string coupling or rank of gauge group.

³As one can show [23], in the case when the classical solution is “sourced” by the vertex operators inserted at the points x_1 and x_2 with $x_1 = -x_2$ one gets the same semiclassical result if one uses the full CPO vertex in [5] or its truncated version in (2.2). We have checked this explicitly for the examples considered below.

get ⁴

$$C_{123} = c_j \sqrt{\lambda} \int_{-\infty}^{\infty} d\tau_e \frac{e^{j\nu\tau_e}}{\cosh^j \kappa\tau_e} (U_1 + U_2) , \quad (2.7)$$

$$U_1 = \frac{2\kappa^2}{\cosh^2 \kappa\tau_e} , \quad U_2 = -(\kappa^2 - \nu^2) . \quad (2.8)$$

We have not used the condition $\kappa^2 = \nu^2$ to illustrate that the result will be *different* if (2.7) is computed using the two different procedures:

- (I) if the on-shell condition $\kappa^2 = \nu^2$ is first used in the integrand;
- (II) if one first does the integral and then takes the limit $\kappa^2 - \nu^2 \rightarrow 0$.

The analog of the U_2 term in (2.8) will always appear if instead of a point-like string we consider a “heavy” state represented by a small string spread in S^5 : then the corresponding Virasoro condition will be $\kappa^2 - \nu^2 = O(\epsilon^2)$ where ϵ is the string size (see below). If one then takes the point-like string limit $\epsilon \rightarrow 0$, one obtains the result which agrees with the C_{123} of the point-like string (2.7), (2.8) computed with the prescription (II). This “anomaly”, i.e. the difference in the outcome of the two prescriptions was first observed in [5] on the example of the folded 2-spin string in the limit when the spin around its center of mass goes to zero. Our aim below will be to expose the universal nature of this “anomaly” and clarify its meaning.

To compute (2.7) we note that the integral over τ can be performed using that

$$\int_0^\infty d\tau_e \frac{\cosh(j\nu\tau_e)}{\cosh^s \kappa\tau_e} = \frac{2^{s-2}}{\kappa \Gamma(s)} \Gamma\left(\frac{s}{2} + \frac{j\nu}{2\kappa}\right) \Gamma\left(\frac{s}{2} - \frac{j\nu}{2\kappa}\right) . \quad (2.9)$$

We then end up with

$$C_{123} = A_1 + A_2 = c_j \sqrt{\lambda} \kappa^{-1} [2\kappa^2 C_1 - (\kappa^2 - \nu^2) C_2] , \quad (2.10)$$

$$C_1 = \frac{2^{j+1}}{\Gamma(j+2)} \Gamma\left(1 + \frac{j}{2} + \frac{j\nu}{2\kappa}\right) \Gamma\left(1 + \frac{j}{2} - \frac{j\nu}{2\kappa}\right) , \quad (2.11)$$

$$C_2 = \frac{j+1}{j} \frac{\kappa^2}{\kappa^2 - \nu^2} C_1 . \quad (2.12)$$

If one uses the first procedure (I) setting $\kappa^2 = \nu^2$ in U_2 in (2.7) then the C_2 contribution is absent. If one uses the second procedure (II), i.e. takes $\kappa^2 - \nu^2 \rightarrow 0$ in the final result (2.10), then the $\kappa^2 - \nu^2$ factor from U_2 term in (2.7) cancels against the pole in C_2 in (2.12). This pole comes from the corresponding Γ -functions in the integral (2.9) and originates from the

⁴Let us comment on the euclidean continuation $\tau \rightarrow \tau_e$. The original integrand has a factor $\frac{1}{\cos \kappa\tau}$ which produces poles on the real axis at $\kappa\tau = \frac{\pi}{2}(2n+1)$, with n =integer. The Wick rotation is performed by choosing a closed contour that does not enclose any pole; for $\text{Re } \tau > 0$ this is done by adding small semi-circles lying on $\text{Im } \tau > 0$ surrounding the poles and closing the contour by a quarter-circle at infinity on the upper plane. Similarly, for $\text{Re } \tau < 0$, the contour is closed by a quarter-circle lying on $\text{Im } \tau < 0$ and adding small semi-circles around the poles lying on $\text{Im } \tau < 0$ in such a way no pole is enclosed. The contributions of the upper and lower quarter-circles vanish because $|\kappa| > \nu$ so that $\frac{e^{ij\nu\tau}}{\cos^j \kappa\tau}$ goes to zero as $\text{Im } \tau \rightarrow \pm\infty$. Thus one is left with the integral over the imaginary axis, i.e. the integral over τ_e from $-\infty$ to ∞ appearing in the above expressions below.

fact that the τ integral multiplying U_2 diverges at $\tau = \infty$ (for $j\nu > 0$) if $\kappa \rightarrow \nu$. Thus there is an additional “anomalous” (or $\frac{0}{0}$) contribution. As a result, we find ($J = \sqrt{\lambda}\nu$)

$$C_{123}^{(I)} = A_1 = c_j \, 2\sqrt{\lambda}(\kappa C_1)_{\kappa=\nu} = c_j \frac{2^{j+2}}{j+1} J, \quad (2.13)$$

$$C_{123}^{(II)} = A_1 + A_2 = \left(1 - \frac{j+1}{2j}\right) C_{123}^{(I)} = \frac{j-1}{2j} C_{123}^{(I)}. \quad (2.14)$$

Using normalization [5, 7, 21] of the CPO vertex given in (2.2), we get for c_j in (2.4)

$$c_j = 2^{-j} \hat{c}_j = \frac{(j+1)\sqrt{j}}{2^{j+2}N}. \quad (2.15)$$

This choice leads to the expected result for the 3-CPO correlator [29] if one uses the first procedure [5]

$$C_{123}^{(I)} = \frac{1}{N} J \sqrt{j} \approx \frac{1}{N} \sqrt{J(J+j)j}. \quad (2.16)$$

While one may argue [5] that the normalization of the CPO vertex is unambiguously fixed by the OPE argument in [21] here we will formally consider the use of the alternative – second – procedure since its result will be smoothly connected with that in the case when the “heavy” state is not BPS, i.e. when one starts with a generic string solution and then takes the string size to zero so that the string state becomes BPS. This is, in fact, how one can understand this second prescription: by starting with a slightly off-BPS expression and taking BPS limit.⁵

As follows from (2.14), in this second case the expression in (2.16) is recovered by changing the normalization in (2.15) to

$$c'_j = \frac{2j}{j-1} c_j = \frac{j(j+1)\sqrt{j}}{2^{j+1}(j-1)N}. \quad (2.17)$$

We thus propose that (2.17) is the right normalization for computing the semiclassical correlators involving at least two non-BPS operators with large charges as well as chiral primary vertex operators.

As will be shown below, it is by adopting this normalization that a general three-point function will reduce to the standard result (2.16) in the limit when non-BPS operators become CPO.⁶ As we shall see below in section 4, the use of this normalization will also simplify subleading (in near BPS limit) terms in the 3-point correlators (the coefficient $\frac{j-1}{2j}$ will appear as an overall factor and will cancel against similar factor in c'_j).

In general, the issue of normalization of the string vertex operator is subtle; an invariant meaning has the ratio of a 3-point function of chiral primary operators (2.1) to their norms,

⁵We are making an implicit assumption that the BPS limit is universal, i.e. does not depend on off-BPS starting point. This appears to be the case as illustrated on explicit examples below.

⁶An alternative possibility, that we shall not pursue here, is that this discontinuity may have a physical meaning and may also have a counterpart on the gauge theory side, being, e.g., related to mixing with double-trace operators (cf. [22]).

and this depends on a prescription of how the 3-point correlator is computed. Since this is an example of an extremal correlator, this computation may be subtle (see below). Indeed, the normalization of the chiral primary operator may be sensitive to the definition of the corresponding supergravity scalars or the choice of basis for the dual gauge theory operators [26].

To argue that the second procedure of computing (2.7) has a natural physical meaning let us note that the $\kappa^2 - \nu^2 = 0$ condition may be interpreted as the marginality condition [4] for the “heavy” vertex operators. In the semiclassical approach this condition may be viewed as originating from extremising with respect to the world-sheet metric (cf. also [3]); in this case it may be imposed at the end of the calculation. Also, since the object we are computing is essentially a scattering amplitude, by analogy with the usual LSZ relation in field theory the mass shell conditions may be implemented in the final expression.

The above $(\frac{\kappa^2 - \nu^2}{\kappa^2 - \nu^2})_{\kappa^2 \rightarrow \nu^2}$ ambiguity is reminiscent of a similar “anomaly” in the computation of extremal $(\Delta_1 = \Delta_2 + \Delta_3)$ 3-point correlators of BPS operators in $AdS_5 \times S^5$. In that case the supergravity 3-point coupling vanishes as $(\Delta_1 - \Delta_2 - \Delta_3) \rightarrow 0$ but it is multiplied by the AdS_5 integral of the three Dirichlet K -functions which is proportional to a product of Γ -functions that contains [24] a pole $\frac{1}{\Delta_1 - \Delta_2 - \Delta_3}$. One can then get the correct non-zero result for the extremal correlator by a formal analytic continuation prescription: first assume that $\Delta_1 = \Delta_2 + \Delta_3 + a$ and then take the limit $a \rightarrow 0$ in the final expression [25]. This prescription may be viewed as formal since for the BPS operators dimensions Δ_i take only integer values⁷ but it becomes more meaningful in the limit of large (semiclassical) charges and, especially, when viewed from the perspective of generic non-BPS string states, i.e., if one defines a correlator of the BPS operators as a limit of a correlator involving non-BPS string states and assumes that the result should be analytic in the string-state quantum numbers.

The 3-point correlator of the chiral primary operators with $\Delta_i = |j_i|$ is always extremal because of the charge conservation. For example, one may choose $j_1 = J$, $j_2 = -(J - j)$, $j_3 = j$ and then the case we considered above corresponds to $J \gg j$. Relaxing the $\kappa = \nu$ condition is the same as moving away from the point $\Delta = |J|$ and is thus similar to relaxing the condition of extremality of the correlator.

As we shall see below, in the computations with non-BPS “heavy” states there will be no similar ambiguity. We shall demonstrate in detail (extending the observation in [5]) that it is the second procedure that is selected if we define the semiclassical correlator (2.4) for BPS “heavy” state as a limit of the correlator where the “heavy” state is represented by a semiclassical string that admits a point-like BPS limit.

3 BPS limit of correlator with “heavy” states as rigid spinning strings on $R \times S^5$

We would like to demonstrate that the result (2.14) of the second procedure discussed in the previous section is indeed a universal outcome of a similar computation for a “heavy state”

⁷Other ways to obtain a consistent non-zero result for the extremal correlators involve accounting for non-linear redefinition of the supergravity scalars related to mixing with double-trace operators on the gauge theory side [26] or by accounting for boundary terms in the supergravity action for the scalar fields [27].

represented by a generic semiclassical string in the subsequent “small-string” or BMN-type limit.

As we will show in this section, this can be done for any solution described by a rigid rotating string ansatz [28].⁸ Rotating strings can have a diversity of rigid shapes, which are determined by the solutions of the Neumann-Rosochatius system. In this section we shall consider the case when the string is point-like in AdS_5 , i.e. is again described by (2.5). Strings extended in AdS_5 will be discussed in section 4.

The S^5 part of the ansatz will be chosen as

$$\begin{aligned} X_1 + iX_2 &= r_1(\sigma)e^{iw_1\tau}, & w_1 &\equiv \nu, \\ X_3 + iX_4 &= r_2(\sigma)e^{i\alpha_2(\sigma)}e^{iw_2\tau}, & X_5 + iX_6 &= r_3(\sigma)e^{i\alpha_3(\sigma)}e^{iw_3\tau}, \end{aligned} \quad (3.1)$$

with ($i = 2, 3$; m_i are integer)

$$\alpha'_i = \frac{v_i}{r_i^2}, \quad \kappa^2 = r_1'^2 + \nu^2 r_1^2 + \sum_{i=2}^3 (r_i'^2 + w_i^2 r_i^2 + \frac{v_i^2}{r_i^2}), \quad (3.2)$$

$$w_2 v_2 + w_3 v_3 = 0, \quad v_i \int_0^{2\pi} \frac{d\sigma}{r_i^2(\sigma)} = 2\pi m_i. \quad (3.3)$$

We have set $\alpha_3 = 0$, that is $v_3 = m_3 = 0$, because we are interested in solutions which smoothly approach the BMN solution

$$v_1 = v_2 = v_3 = 0, \quad r_1 = 1, \quad \kappa = \nu. \quad (3.4)$$

The equations of motion for $r_s(\sigma)$ are

$$r_1'' + (\nu^2 + \Lambda)r_1 = 0, \quad r_i'' - \frac{v_i^2}{r_i^3} + (w_i^2 + \Lambda)r_i = 0, \quad i = 2, 3 \quad (3.5)$$

$$\sum_{s=1}^3 r_s^2 = 1, \quad \Lambda = \Lambda(\sigma) = \sum_{s=1}^3 (r_s'^2 + \frac{v_s^2}{r_s^2} - w_s^2 r_s^2). \quad (3.6)$$

Let us now consider this solution (which, in general, carries three $SO(6)$ spins J_1, J_2, J_3) as representing the “heavy” state in the computation of the 3-point function in (2.4). Using (2.2) we obtain the following generalization of (2.7),(2.8) (assuming again that $\tau = -i\tau_e$)

$$C_{123} = \sqrt{\lambda} c_j \int_{-\infty}^{\infty} d\tau_e \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[\frac{r_1(\sigma)e^{\nu\tau_e}}{\cosh \kappa\tau_e} \right]^j [U_1(\tau_e) + U_2(\sigma)], \quad (3.7)$$

$$U_1 = \frac{2\kappa^2}{\cosh^2 \kappa\tau_e}, \quad U_2 = -\kappa^2 - \partial^a X_k \partial_a X_k = -\kappa^2 - \Lambda(\sigma), \quad (3.8)$$

where we used (3.6). Since Λ and thus U_2 do not depend on τ_e , the integral over τ_e in (3.7) can be performed in the same way as in the previous section (i.e. using (2.9)) and we get the

⁸Such strings are “stationary”, i.e. there exists a rotating frame in which they are static.

expression generalizing (2.10)

$$C_{123} = A_1 + A_2 , \quad (3.9)$$

$$A_1 = 2\sqrt{\lambda}\kappa \, c_j \, C_1 \int_0^{2\pi} \frac{d\sigma}{2\pi} [r_1(\sigma)]^j , \quad (3.10)$$

$$A_2 = -\sqrt{\lambda}\kappa \, c_j \, C_2 \int_0^{2\pi} \frac{d\sigma}{2\pi} [r_1(\sigma)]^j \left(1 + \frac{\Lambda}{\kappa^2}\right) , \quad (3.11)$$

where C_1 and C_2 are the same as in (2.11),(2.12).

Let us now expand around the BMN solution (3.4) for which $\Lambda \rightarrow -\nu^2$. In this case

$$\Lambda = -\nu^2 + \epsilon^2 \tilde{\Lambda} , \quad \epsilon \rightarrow 0 , \quad (3.12)$$

$$r_2 = \epsilon \tilde{r}_2 , \quad r_3 = \epsilon \tilde{r}_3 , \quad r_1 = \sqrt{1 - \epsilon^2(\tilde{r}_2^2 + \tilde{r}_3^2)} , \quad (3.13)$$

$$v_2 = \epsilon^2 \tilde{v}_2 , \quad v_3 = \epsilon^2 \tilde{v}_3 , \quad \kappa^2 = \nu^2 + \epsilon^2 \tilde{\kappa}^2 , \quad (3.14)$$

where we assume that the variables with tilde are of order 1. Such solution describes small strings of rigid shape rotating on S^5 with $J_1 = \sqrt{\lambda}\nu \gg J_2, J_3$.

Using (3.12)-(3.14) and expanding in powers of ϵ , we find from (2.11),(3.10)

$$C_1 = \frac{2^{j+1}}{j+1} + O(\epsilon^2) , \quad (3.15)$$

$$A_1 = c_j(j+1)^{-1}2^{j+2}J_1 + O(\epsilon^2) , \quad J_1 = \sqrt{\lambda} \, \nu \quad (3.16)$$

with the leading term being the same as in (2.13). To compute the second contribution A_2 in (3.11) coming from the U_2 term in (3.8) we note that according to (2.11),(3.12) (3.14)

$$\kappa\left(1 + \frac{\Lambda}{\kappa^2}\right) = \frac{\tilde{\kappa}^2 + \tilde{\Lambda}}{\nu} \epsilon^2 + O(\epsilon^4) , \quad (3.17)$$

$$C_2 = \frac{2^{j+1}\nu^2}{j\tilde{\kappa}^2} \frac{1}{\epsilon^2} + O(1) . \quad (3.18)$$

This leads to a finite “anomalous” contribution

$$A_2 = -c_j j^{-1} J_1 \left(1 + \frac{1}{\tilde{\kappa}^2} \int_0^{2\pi} \frac{d\sigma}{2\pi} \tilde{\Lambda}\right) + O(\epsilon^2) . \quad (3.19)$$

To compute the integral over σ we use (3.5),(3.13), i.e.

$$r_1'' + \epsilon^2 \tilde{\Lambda} r_1 = 0 , \quad r_1 = 1 + O(\epsilon^2) , \quad (3.20)$$

$$\tilde{\Lambda} = -\frac{r_1''}{r_1 \epsilon^2} = (\tilde{r}_2 \tilde{r}_2' + \tilde{r}_3 \tilde{r}_3')' + O(\epsilon^2) . \quad (3.21)$$

Thus $\tilde{\Lambda}$ is a total derivative which drops out of the integral over σ in (3.19); in the limit $\epsilon \rightarrow 0$ we are left with

$$A_2 = -c_j j^{-1} 2^{j+1} J_1 + O(\epsilon^2) = -\frac{j+1}{2j} A_1 + O(\epsilon^2) , \quad (3.22)$$

which is the direct analog of the expression for the second “anomalous” contribution in (2.14). We observe that this “anomalous” contribution is universal for all rigid string solutions on S^5 which approach the BMN one (i.e. it does not depend on the shape of the string $r_i(\sigma)$ or parameters v_i).

Altogether, we find the same expression for (3.7) as in (2.14), i.e. as in the second procedure of section 3,

$$C_{123} = \frac{j-1}{2j} A_1 + O(\epsilon^2) . \quad (3.23)$$

This leads us to the standard 3-CPO result (2.16) if we adopt the modified normalization in (2.17), $c_j \rightarrow c'_j$, i.e. $C_{123} \rightarrow C'_{123}$,

$$\lim_{\epsilon \rightarrow 0} C'_{123} = \frac{1}{N} J_1 \sqrt{j} , \quad C'_{123} \equiv C_{123}(c_j \rightarrow c'_j) . \quad (3.24)$$

We conclude that to have a smooth BPS limit, the 3-point function involving “heavy” states with generic values of the spins (J_1, J_2, J_3) should be computed with the normalization (2.17) for the “light” chiral primary vertex operator.

4 Large spin expansion of correlators with “heavy” states corresponding to spinning strings in $AdS_5 \times S^5$

Our aim in this section will be to consider several explicit examples involving “heavy” states represented by spinning strings in $AdS_5 \times S^5$ that admit a point-like (BMN-type) large J limit and to compute subleading $1/J$ terms in the 3-point coefficient C_{123} . We shall see that these subleading terms have the same structure as expected in perturbative gauge theory, suggesting that their coefficients are protected, i.e. should match the gauge theory ones (see the discussion in the Introduction).

The examples we will consider below are: folded spinning string on S^5 (with spins J_1, J_2), folded spinning string extended on AdS_5 and orbiting S^5 (with spins S, J), and a circular string extended and rotating on both AdS_5 and S^5 (with spins S, J_1, J_2, J_3).

4.1 Folded (J_1, J_2) spinning string in S^5

The solution describing a folded string carrying angular momenta J_1 and J_2 is a particular example of a rigid string solution on S^5 discussed in the previous section. In this case we have (3.1) with $\alpha_i = 0$, $r_3 = 0$ and [13]

$$r_1 = \cos \theta = \text{dn}(w_{21}\sigma|q) , \quad r_2 = \sin \theta = \sqrt{q} \, \text{sn}(w_{21}\sigma|q) , \quad (4.1)$$

$$q \equiv \sin^2 \theta_0 = \frac{\kappa^2 - w_1^2}{w_2^2 - w_1^2} , \quad w_{21}^2 \equiv w_2^2 - w_1^2 = \frac{2}{\pi} K(q) . \quad (4.2)$$

Assuming that the “heavy” state in the 3-point correlator (2.4) is represented by such string carrying two angular momenta (J_1, J_2) , we find from (3.7)⁹

$$C'_{123} = 2\sqrt{\lambda} \, c'_j \int_{-\infty}^{\infty} d\tau_e \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[\frac{\cos \theta(\sigma) e^{w_1 \tau_e}}{\cosh \kappa \tau_e} \right]^j \left[\frac{\kappa^2}{\cosh^2 \kappa \tau_e} - \kappa^2 - w_{21}^2 \cos^2 \theta(\sigma) + w_2^2 \right]. \quad (4.3)$$

This example was previously considered in [5]. Here we put prime on C_{123} since we will use the modified normalization c'_j of the “light” chiral primary operator in (2.17) to get a smooth BMN-type limit when $J_1 \gg J_2$ as discussed in the previous section. We will be interested in the explicit form of the subleading corrections to the leading CPO term (2.16).

According to (3.9)–(3.11) we get $C'_{123} = A_1 + A_2$ with

$$A_1 = 8\sqrt{\lambda} \, \kappa \, c'_j \, C_1 \int_0^{\frac{\pi}{2}} \frac{d\sigma}{2\pi} [\cos \theta(\sigma)]^j, \quad (4.4)$$

$$A_2 = -8\sqrt{\lambda} \, \kappa^{-1} \, c'_j \, C_2 \int_0^{\frac{\pi}{2}} \frac{d\sigma}{2\pi} [\cos \theta(\sigma)]^j [\kappa^2 + w_{21}^2 \cos^2 \theta(\sigma) - w_2^2], \quad (4.5)$$

where C_1 and C_2 as in (2.11),(2.12) with $\nu = w_1$. We took into account that for folded string the σ integration region is naturally split into 4 equivalent parts.

Defining

$$\mathcal{J}_i = \frac{J_i}{\sqrt{\lambda}}, \quad \varepsilon = \frac{\mathcal{J}_2}{\mathcal{J}}, \quad \mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2, \quad (4.6)$$

we have $\varepsilon \ll 1$ in the near-BMN limit when $\mathcal{J}_1 \gg \mathcal{J}_2$. We shall first expand in powers of $\frac{1}{\mathcal{J}_i}$ and then expand each coefficient of this expansion in powers of ε . In doing so one finds [13]

$$w_1 = \mathcal{J} - \frac{\varepsilon}{2\mathcal{J}}(1 + \varepsilon + \dots) + \dots, \quad w_2 = \mathcal{J} + \frac{1}{2\mathcal{J}}(1 + \frac{\varepsilon^2}{8} + \dots) + \dots, \quad (4.7)$$

$$\kappa = \mathcal{J} + \frac{\varepsilon}{2\mathcal{J}}(1 + \frac{\varepsilon}{2} + \dots) + \dots, \quad q = 2\varepsilon(1 - \frac{\varepsilon}{4} - \frac{\varepsilon^2}{8} + \dots), \quad (4.8)$$

$$\theta(\sigma) = \sqrt{2} \left(\varepsilon^{1/2} \sin(\sigma) + \frac{1}{24} \varepsilon^{3/2} [6 \sin(\sigma) + \sin(3\sigma)] + \dots \right). \quad (4.9)$$

Then the second factor in the integrand of A_2 in (4.5) has the following expansion in small $\varepsilon = \frac{\mathcal{J}_2}{\mathcal{J}}$

$$\kappa^2 + w_{21}^2 \cos^2 \theta(\sigma) - w_2^2 = \left[2 \cos^2 \sigma - \frac{1}{4\mathcal{J}^2} (1 + 2 \sin^2 \sigma) \right] \varepsilon + O(\varepsilon^2). \quad (4.10)$$

As in the general case of a rigid spinning string (3.17), it vanishes in the point-like limit ($\mathcal{J}_2 \rightarrow 0$). Using (2.12), we find that as in (3.15),(3.18)

$$C_1 = \frac{2^{j+1}}{j+1} + O(\varepsilon), \quad C_2 = \frac{2^j}{j} \frac{\mathcal{J}^2}{\varepsilon} + O(1). \quad (4.11)$$

⁹We use that here Λ in (3.6) is $\Lambda = r_1'^2 + r_2'^2 - w_1^2 r_1^2 - w_2^2 r_2^2$, while the conformal gauge constraint in (3.2) takes the form $\kappa^2 = r_1'^2 + r_2'^2 + w_1^2 r_1^2 + w_2^2 r_2^2$.

Thus we obtain the same expressions (3.22),(3.23),(3.24) as the leading term in the $\mathcal{J}_2 \rightarrow 0$ or $\varepsilon \rightarrow 0$ limit.

Including corrections in $\frac{\mathcal{J}_2}{\mathcal{J}}$, we observe that the ratio $\frac{i-1}{2j}$ factors out also from the subleading terms and is thus cancelled against a similar factor in the normalization coefficient c'_j in (2.17). We thus end up with ¹⁰

$$C'_{123} = \frac{1}{N} \sqrt{\lambda} \mathcal{J} \sqrt{j} \left[1 - \frac{\varepsilon}{4} (1 + 3j) + O(\varepsilon^2) + \frac{\varepsilon}{8\mathcal{J}^2} \left(5 + 3j - 4j(\psi(j) + \gamma) + O(\varepsilon) \right) + O\left(\frac{\varepsilon}{\mathcal{J}^4}\right) \right], \quad (4.12)$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ and γ is the Euler's constant. Equivalently, in terms of the spins J_1, J_2 , and λ this reads, in the limit when $J_1 \gg J_2$,

$$C'_{123} = \frac{1}{N} J_1 \sqrt{j} \left[1 - \frac{3J_2}{4J_1} (j - 1) - \lambda \frac{J_2}{8J_1^3} \left(4j(\psi(j) + \gamma - \frac{3}{4}) - 5 \right) + O\left(\frac{J_2^2}{J_1^2}\right) + O\left(\frac{\lambda J_2^2}{J_1^4}\right) + O\left(\frac{\lambda^2 J_2}{J_1^5}\right) \right]. \quad (4.13)$$

Observing that the charges of the three operators entering the correlator may be assumed to be $(J_1, J_2), (-J_1 - j, -J_2), (j, 0)$ we conclude that for $J_1 \gg J_2 \gg j$ the leading term is the standard protected 3-CPO correlator.

The two explicitly written subleading terms in (4.13) look like the tree-level and the one-loop corrections on gauge theory side.¹¹ We conjecture that their coefficients are also not renormalized, i.e. they should be the same in the perturbative string ($\lambda \gg 1$) and in the perturbative gauge ($\sqrt{\lambda} \ll 1$) theories.¹² It thus remains to compare (4.13) in detail with the one-loop $\mathcal{N}=4$ SYM correlator of the corresponding $SU(2)$ sector operators in the limit $J_1 \gg J_2 \gg j$.

4.2 Folded (S, J) spinning string in $AdS_5 \times S^1$

Let us now repeat the same computation in the case when the “heavy” state is represented by the folded spinning string in AdS_3 part of AdS_5 orbiting big circle of S^5 [30, 31]. In the limit of large spin this example was already discussed in [7]. For generic values of spins (S, J) the solution written in global AdS_5 coordinates is

$$\begin{aligned} \rho &= \rho(\sigma), \quad t = \kappa\tau, \quad \phi = w\tau, \quad \varphi = \nu\tau, \quad \rho'^2 = \kappa^2 \cosh^2 \rho - w^2 \sinh^2 \rho - \nu^2, \\ \sinh \rho &= \epsilon \operatorname{sn}(\sqrt{\kappa^2 - \nu^2} \epsilon^{-1} \sigma, -\epsilon^2), \quad \epsilon^2 \equiv \frac{\kappa^2 - \nu^2}{w^2 - \kappa^2}, \quad \sqrt{\kappa^2 - \nu^2} = \epsilon {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; -\epsilon^2\right) \end{aligned} \quad (4.14)$$

¹⁰This is, of course, the leading semiclassical approximation, i.e. we ignore string $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ corrections that lead to terms suppressed by extra powers of $\frac{1}{j} = \frac{1}{\sqrt{\lambda}} \frac{1}{\mathcal{J}}$.

¹¹For example, the 1-loop correlator of 3 near-BMN operators with large charges $-J, rJ$ and $(1-r)J$ has the following form [19]: $C_{123} = \frac{1}{N} \sqrt{J^3 r(1-r)} \left(a_1 + a_2 \frac{\lambda}{\mathcal{J}^2} + \dots \right)$ where a_1 and a_2 depend on detailed structure of the operators (number of impurities, etc.).

¹²It is likely that the same non-renormalization should apply also to the coefficient of the next term we did not write explicitly which is proportional to λ^2 .

$\rho(\sigma)$ varies from 0 up to its maximal value given by $\coth^2 \rho_{\max} = \frac{w^2 - \nu^2}{\kappa^2 - \nu^2} = 1 + \frac{1}{\epsilon^2}$ and the angular momenta are

$$S = \sqrt{\lambda} \mathcal{S} = \frac{\sqrt{\lambda} w \epsilon^2}{2\sqrt{\kappa^2 - \nu^2}} {}_2F_1\left(\frac{1}{2}, \frac{3}{2}, 2; -\epsilon^2\right), \quad J = \sqrt{\lambda} \mathcal{J}, \quad \mathcal{J} = \nu. \quad (4.15)$$

Written in the Poincaré coordinates and continued to the euclidean time the AdS_5 part of the solution is [4, 7] (cf. (2.5))

$$Y_4 = 0, \quad Y_5 = \frac{1}{z} = \cosh \rho \cosh \kappa \tau_e, \quad Y_{0e} = \frac{x_{0e}}{z} = \cosh \rho \sinh \kappa \tau_e, \quad (4.16)$$

$$z = \frac{1}{\cosh \rho \cosh \kappa \tau_e}, \quad x_{0e} = \tanh \kappa \tau_e, \quad x_1 + ix_2 = \frac{\tanh \rho}{\cosh \kappa \tau_e} e^{w \tau_e}. \quad (4.17)$$

For the 3-point correlator of the states with charges $(S, J), (-S, -J - j)$ and $(0, j)$ where $J \gg j$ we find from (2.4) the following analog of the expression (4.3)

$$C'_{123} = 8\sqrt{\lambda} c'_j \int_{-\infty}^{\infty} d\tau_e \int_0^{\frac{\pi}{2}} \frac{d\sigma}{2\pi} \left[\frac{e^{\nu \tau_e}}{\cosh \rho(\sigma) \cosh \kappa \tau_e} \right]^j \left[\frac{\kappa^2}{\cosh^2 \kappa \tau_e} - (w^2 - \nu^2) \tanh^2 \rho(\sigma) \right] \quad (4.18)$$

Let us consider the short string limit when $S \ll J$, i.e. $\epsilon \rightarrow 0$ and $\rho_{\max} = \epsilon + O(\epsilon^3)$ so that ϵ measures the size of the string. We will expand (4.18) in powers of ϵ , i.e. approach the BMN limit from the $\epsilon \neq 0$ region. As in the previous sections, we will then reproduce the 3-CPO result (3.24) if we use normalisation constant in (2.17).

Explicitly, expanding in ϵ we get

$$\rho = \epsilon \sin \sigma - \frac{1}{12} \epsilon^3 (3 \sin \sigma - \sin^3 \sigma) + O(\epsilon^5), \quad \epsilon^2 = \frac{2\mathcal{S}}{\mathcal{J}} + \frac{\mathcal{S}^2}{2\mathcal{J}^2} + \dots \quad (4.19)$$

$$\kappa^2 = \nu^2 + \epsilon^2 - \frac{\epsilon^4}{2} + \dots, \quad w^2 = 1 + \nu^2 + \frac{\epsilon^2}{2} + \dots. \quad (4.20)$$

Computing the integral over τ_e we get $C'_{123} = A_1 + A_2$ with (cf. (3.10), (3.11) and (4.4), (4.5))

$$\begin{aligned} A_1 &= 8\sqrt{\lambda} \kappa c'_j C_1 \int_0^{\frac{\pi}{2}} \frac{d\sigma}{2\pi} \frac{1}{\cosh^j \rho(\sigma)}, \\ A_2 &= -8\sqrt{\lambda} c'_j C_2 \kappa^{-1} (w^2 - \nu^2) \int_0^{\frac{\pi}{2}} \frac{d\sigma}{2\pi} \frac{\sinh^2 \rho(\sigma)}{\cosh^{j+2} \rho(\sigma)}, \end{aligned} \quad (4.21)$$

where C_1 and C_2 are again given in (2.11), (2.12). Expanding in powers of ϵ , we find that, again, $C'_{123} \sim \frac{j-1}{2j} c'_j$ with $\frac{j-1}{2j}$ factorising out also of the subleading terms. As a result, we obtain

$$C'_{123} = \frac{1}{N} J \sqrt{j} \left[1 - \frac{\epsilon^2 (j-1)}{8} - \frac{\epsilon^2}{4\mathcal{J}^2} (j\psi(j) + j\gamma - 1) + O(\epsilon^4) + O\left(\frac{\epsilon^4}{\mathcal{J}^2}\right) \right], \quad (4.22)$$

or, equivalently,

$$\begin{aligned} C'_{123} &= \frac{1}{N} J \sqrt{j} \left[1 - \frac{S}{4J} (j-1) - \lambda \frac{S}{2J^3} (j(\psi(j) + \gamma) - 1) \right. \\ &\quad \left. + O\left(\frac{S^2}{J^2}\right) + O\left(\frac{\lambda S^2}{J^4}\right) + O\left(\frac{\lambda^2 S^2}{J^6}\right) \right]. \end{aligned} \quad (4.23)$$

This is very similar in structure to the expression (4.13) in the case of the (J_1, J_2) folded string. Note that there is no other $O(\frac{1}{J^n})$ correction to the $O(\epsilon^2)$ term in (4.22), i.e. the next subleading term comes from $O(\frac{\epsilon^4}{J^4})$, i.e. is $O(\frac{\lambda^2 S^2}{J^6})$.

Once again, we conjecture that the leading terms in (4.23) should match the leading terms in the 3-point correlator of the one-loop $\mathcal{N}=4$ SYM correlator of the corresponding $SL(2)$ sector operators with charges (S, J) , $(-S, -J - j)$ and $(0, j)$ in the limit $J \gg S$, $J \gg j$.

4.3 Circular spinning string in $AdS_5 \times S^5$

Another example involves a rigid circular string with spin S in one plane in AdS_5 and spins J_1, J_2, J_3 in the three orthogonal planes of S^5 . It allows us to provide another test of the universality of the “anomaly” (3.23) and thus of the modified normalization (2.17) in the case when the string is extended in both AdS_5 and S^5 directions.¹³

This is a solution of the general $AdS_5 \times S^5$ rigid string ansatz (cf. (3.1)) in which the radii r_i are constant, i.e. it represents a circular string which winds and rigidly rotates in several planes [28]

$$Y_5 + iY_0 = b_0 e^{i\kappa\tau}, \quad Y_1 + iY_2 = b_1 e^{ik\sigma} e^{i\omega_1\tau}, \quad Y_3 + iY_4 = 0, \quad (4.24)$$

$$X_1 + iX_2 = a_1 e^{iw_1\tau}, \quad X_3 + iX_4 = a_2 e^{im_2\sigma} e^{iw_2\tau}, \quad X_5 + iX_6 = a_3 e^{im_3\sigma} e^{iw_3\tau} \quad (4.25)$$

$$w_1 = \nu, \quad a_1^2 + a_2^2 + a_3^2 = 1, \quad b_0^2 - b_1^2 = 1, \quad (4.26)$$

$$kS + m_2 J_2 + m_3 J_3 = 0, \quad S = \sqrt{\lambda} \mathcal{S}, \quad J_i = \sqrt{\lambda} \mathcal{J}_i. \quad (4.27)$$

Here k and m_i are integer winding numbers. We have set $m_1 = 0$ to have a smooth BMN limit when the string becomes point-like. For simplicity we ignore the possibility of the second spin component in AdS_5 .

Computing (2.4) in this case we find (cf.(2.7) and (3.7) where now $\Lambda = -\nu^2$)

$$C'_{123} = \sqrt{\lambda} \, c'_j \left(\frac{a_3}{b_0} \right)^j \int_{-\infty}^{\infty} d\tau_e \frac{e^{j\nu\tau_e}}{\cosh^j \kappa\tau_e} \left(\frac{2\kappa^2}{\cosh^2 \kappa\tau_e} - \kappa^2 + \nu^2 \right). \quad (4.28)$$

As in the previous sections, we get $C'_{123} = A_1 + A_2$, with

$$A_1 = 2\sqrt{\lambda} \, \kappa \, c'_j \left(\frac{a_3}{b_0} \right)^j C_1, \quad A_2 = -\sqrt{\lambda} \, \kappa^{-1} \, c'_j \left(\frac{a_3}{b_0} \right)^j C_2 (\kappa^2 - \nu^2) \quad (4.29)$$

where C_1, C_2 are as in (2.11), (2.12). Note that this is formally the same as (2.10) except for the factor $(\frac{a_3}{b_0})^j$. In the limit when $J_1 \gg J_2, J_3, S$, we have that $\kappa \rightarrow \nu$. Then, once again, the pole in C_2 (2.12) cancels against the zero of the numerator in (4.29), producing an extra finite contribution as compared to the strict BMN limit where $\kappa = \nu$ is imposed before the integration over τ_e .

¹³The corresponding 3-point function was recently computed also in the first paper in ref. [8] but the fact that taking the BMN limit in the final expression leads to an additional “anomalous” contribution and thus requires to use the normalization (2.17) was not noticed there.

Assuming that $\mathcal{J}_i, \mathcal{S} \gg m_i, k$ we get the following expansions (see [28] for details):

$$\begin{aligned}\nu &= \mathcal{J} - \frac{1}{2\mathcal{J}^2} (\mathcal{J}_2 m_2^2 + \mathcal{J}_3 m_3^2) + \frac{1}{8\mathcal{J}^5} [3\mathcal{J}(\mathcal{J}_2 m_2^4 + \mathcal{J}_3 m_3^4) - 4(\mathcal{J}_2 m_2^2 + \mathcal{J}_3 m_3^2)^2] + O(\mathcal{J}^{-5}), \\ \kappa &= \mathcal{J} + \frac{1}{2\mathcal{J}^2} (\mathcal{J}_2 m_2^2 + \mathcal{J}_3 m_3^2 + 2\mathcal{S}k^2) \\ &\quad - \frac{1}{8\mathcal{J}^5} [\mathcal{J}(\mathcal{J}_2 m_2^4 + \mathcal{J}_3 m_3^4 + 4\mathcal{S}k^2) + 4k^2\mathcal{S}(2\mathcal{J}_2 m_2^2 + 2\mathcal{J}_3 m_3^2 + 3\mathcal{S}k^2)] + O(\mathcal{J}^{-5}),\end{aligned}\quad (4.30)$$

where $\mathcal{J} \equiv \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 = \frac{J}{\sqrt{\lambda}}$. Then we find from (4.29) that again the overall $\frac{i-1}{2j}$ factor cancels against the normalization factor c'_j in (2.17), i.e.

$$C'_{123} = \frac{1}{N} J \sqrt{j} \left(\frac{J_1}{J+S} \right)^{j/2} \left(1 + \frac{\lambda}{2J^3} P_1 + O\left(\frac{\lambda^2}{J^4}\right) \right), \quad (4.31)$$

$$P_1 = k^2 S - \left(J_2 m_2^2 + J_3 m_3^2 + k^2 S \right) \left(j(\psi(j) + \gamma - 1) - 1 + \frac{jJ}{2(J+S)} \right). \quad (4.32)$$

The factor $\left(\frac{J_1}{J+S}\right)^{j/2}$ originates from $\left(\frac{a_3}{b_0}\right)^j$. Note that this expansion applies for general (large) values of J_1, J_2, J_3, S , i.e. so far we did not assume a near-BMN limit.

In the near-BMN or point-like string limit $J_1 \gg J_2, J_3, S$ (when $a_3, b_0 \rightarrow 1$, $\left(\frac{J_1}{J+S}\right)^{j/2} \rightarrow 1$) the expression (4.31) for C'_{123} takes the form similar to the one in (4.13) and (4.23) with the leading term in the “one-loop” coefficient P_1 simplifying to

$$P_1^{(1)} = k^2 S - \left(J_2 m_2^2 + J_3 m_3^2 + k^2 S \right) \left(j(\psi(j) + \gamma - \tfrac{1}{2}) - 1 \right). \quad (4.33)$$

As in the previous examples (4.13), (4.23) the leading term in (4.31) is then the 3-CPO correlator and subleading terms look like tree-level and one-loop gauge-theory terms expanded in the limit $J_1 \gg J_2, J_3, S \gg m_i, k$. Again, it would be very interesting to compare these expressions to the gauge theory ones for the correlator of the corresponding operators with charges $(S; J_1, J_2, J_3)$, $(-S; -J_1 - j, -J_2, -J_3)$ and $(0; j, 0, 0)$.

Acknowledgements

We are grateful to E. Buchbinder and R. Roiban for collaboration on related problems. We also acknowledge R. Roiban and K. Zarembo for helpful remarks on the draft. We also thank F. Alday, N. Gromov, G. Korchemsky and P. Vieira for discussions. J.R. acknowledges support by MCYT Research Grant No. FPA 2007-66665 and Generalitat de Catalunya under project 2009SGR502.

References

- [1] D. Serban, “Integrability and the AdS/CFT correspondence,” [arXiv:1003.4214].
- [2] E. I. Buchbinder, “Energy-Spin Trajectories in $AdS_5 \times S^5$ from Semiclassical Vertex Operators,” JHEP **1004**, 107 (2010). [arXiv:1002.1716].
- [3] R. A. Janik, P. Surowka and A. Wereszczynski, “On correlation functions of operators dual to classical spinning string states,” JHEP **1005**, 030 (2010). [arXiv:1002.4613].
- [4] E. I. Buchbinder and A. A. Tseytlin, “On semiclassical approximation for correlators of closed string vertex operators in AdS/CFT,” JHEP **1008**, 057 (2010). [arXiv:1005.4516].
- [5] K. Zarembo, “Holographic three-point functions of semiclassical states,” JHEP **1009**, 030 (2010). [arXiv:1008.1059].
- [6] M. S. Costa, R. Monteiro, J. E. Santos and D. Zaokas, “On three-point correlation functions in the gauge/gravity duality,” JHEP **1011**, 141 (2010). [arXiv:1008.1070].
- [7] R. Roiban and A. A. Tseytlin, “On semiclassical computation of 3-point functions of closed string vertex operators in $AdS_5 \times S^5$,” Phys. Rev. D **82**, 106011 (2010). [arXiv:1008.4921].
- [8] R. Hernandez, “Three-point correlation functions from semiclassical circular strings,” [arXiv:1011.0408]. S. Ryang, “Correlators of Vertex Operators for Circular Strings with Winding Numbers in $AdS_5 \times S^5$,” [arXiv:1011.3573]. G. Georgiou, “Two and three-point correlators of operators dual to folded string solutions at strong coupling,” arXiv:1011.5181.
- [9] A. A. Tseytlin, “On semiclassical approximation and spinning string vertex operators in $AdS_5 \times S^5$,” Nucl. Phys. B **664**, 247 (2003) [hep-th/0304139].
- [10] D. Chialva, R. Iengo and J. G. Russo, “Search for the most stable massive state in superstring theory,” JHEP **0501**, 001 (2005) [arXiv:hep-th/0410152].
- [11] S. Frolov and A. A. Tseytlin, “Multi-spin string solutions in $AdS_5 \times S^5$,” Nucl. Phys. B **668**, 77 (2003) [hep-th/0304255].
- [12] N. Beisert, J. A. Minahan, M. Staudacher and K. Zarembo, “Stringing spins and spinning strings,” JHEP **0309**, 010 (2003) [hep-th/0306139].
- [13] S. Frolov and A. A. Tseytlin, “Rotating string solutions: AdS/CFT duality in non-supersymmetric sectors,” Phys. Lett. B **570**, 96 (2003) [arXiv:hep-th/0306143].
- [14] N. Beisert, S. Frolov, M. Staudacher and A.A. Tseytlin, “Precision spectroscopy of AdS / CFT,” JHEP **0310**, 037 (2003). [hep-th/0308117].
- [15] N. Beisert, A. A. Tseytlin, “On quantum corrections to spinning strings and Bethe equations,” Phys. Lett. B **629**, 102-110 (2005). [hep-th/0509084].
- [16] N. Beisert, B. Eden, M. Staudacher, “Transcendentality and Crossing,” J. Stat. Mech. **0701**, P021 (2007). [hep-th/0610251].
- [17] S. Giombi, R. Ricci, R. Roiban and A. A. Tseytlin, “Two-loop adss superstring: testing asymptotic Bethe ansatz and finite size corrections,” arXiv:1010.4594.

- [18] K. Okuyama and L. S. Tseng, “Three-point functions in $N = 4$ SYM theory at one-loop,” JHEP **0408**, 055 (2004) [hep-th/0404190]. L. F. Alday, J. R. David, E. Gava, and K. Narain, “Structure constants of planar $N = 4$ Yang Mills at one loop,” JHEP **0509**, 070 (2005). [hep-th/0502186]. A. Grossardt and J. Plefka, “One-Loop Spectroscopy of Scalar Three-Point Functions in planar $N=4$ super Yang-Mills Theory,” arXiv:1007.2356. R. Roiban, A. Volovich, “Yang-Mills correlation functions from integrable spin chains,” JHEP **0409**, 032 (2004). [hep-th/0407140].
- [19] N. Beisert, C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, “BMN correlators and operator mixing in $N = 4$ super Yang-Mills theory,” Nucl. Phys. B **650**, 125 (2003) [arXiv:hep-th/0208178]. N. R. Constable, D. Z. Freedman, M. Headrick and S. Minwalla, “Operator mixing and the BMN correspondence,” JHEP **0210**, 068 (2002) [arXiv:hep-th/0209002]. C. Kristjansen, “Review of AdS/CFT integrability, Chapter IV.1: Aspects of non-planarity, arXiv:1012.3997.
- [20] J. Escobedo, N. Gromov, A. Sever and P. Vieira, “Tayloring three-point functions and integrability”, arXiv:1012.2475.
- [21] D. E. Berenstein, R. Corrado, W. Fischler and J. Maldacena, “The Operator product expansion for Wilson loops and surfaces in the large N limit,” Phys. Rev. **D59**, 105023 (1999). [hep-th/9809188].
- [22] G. Georgiou, V. L. Gili and R. Russo, “Operator mixing and three-point functions in $N=4$ SYM,” JHEP **0910**, 009 (2009). [arXiv:0907.1567].
- [23] E. I. Buchbinder, A. A. Tseytlin, “Semiclassical four-point functions in $AdS_5 \times S^5$,” [arXiv:1012.3740].
- [24] D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, “Correlation functions in the CFT(d)/AdS(d+1) correspondence,” Nucl. Phys. **B546**, 96-118 (1999). [hep-th/9804058].
- [25] H. Liu and A. A. Tseytlin, “Dilaton - fixed scalar correlators and $AdS_5 \times S^5$ - SYM correspondence,” JHEP **9910**, 003 (1999). [hep-th/9906151].
- [26] G. Arutyunov and S. Frolov, “Some cubic couplings in type IIB supergravity on $AdS_5 \times S^5$ and three point functions in SYM(4) at large N ,” Phys. Rev. **D61**, 064009 (2000). [hep-th/9907085]; “Scalar quartic couplings in type IIB supergravity on $AdS_5 \times S^5$,” Nucl. Phys. **B579**, 117-176 (2000). [hep-th/9912210]; “On the correspondence between gravity fields and CFT operators,” JHEP **0004**, 017 (2000). [hep-th/0003038].
- [27] E. D’Hoker, D. Z. Freedman, S. D. Mathur and L. Rastelli, “Extremal correlators in the AdS / CFT correspondence,” In: Shifman, M.A. (ed.): The many faces of the superworld, 332-360. [hep-th/9908160].
- [28] G. Arutyunov, J. Russo and A. A. Tseytlin, “Spinning strings in $AdS_5 \times S^5$: New integrable system relations,” Phys. Rev. D **69**, 086009 (2004) [arXiv:hep-th/0311004].
- [29] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, “Three-point functions of chiral operators in $D = 4$, $N = 4$ SYM at large N ,” Adv. Theor. Math. Phys. **2**, 697 (1998) [arXiv:hep-th/9806074].
- [30] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “A semi-classical limit of the gauge/string correspondence,” Nucl. Phys. B **636**, 99 (2002), [hep-th/0204051].
- [31] S. Frolov and A. A. Tseytlin, “Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$,” JHEP **0206**, 007 (2002), [hep-th/0204226].